Lesson 44

# Math 70 Quadratic Functions and their Graphs

GOAL: Graph parabolas neatly and accurately on paper. This means:

1) Imagine the location, shape, and direction and calculate the vertex to determine axes and scales.
2) Plot the vertex. GC max or min can help.
3) Plot four additional points – might need to move the axes. GC table can help.
4) The pattern of squares can accurately plot points without doing *any* calculations, if both scales are 1.
5) Confirm that the left side and right side are symmetric.
6) Confirm that your graph extends to the edges of the grid accurately – might need more than five points.
7) Confirm that your parabola is round as it passes through the vertex.
8) If the instructions request it: draw the axis of symmetry as a dotted vertical line through the vertex.
9) If the instructions say: find the *y*-intercept and plot it.
10) If the instructions say: find the *x*-intercepts and plot them.

To begin, quadratic functions (up- or downward opening parabolas) are in the form $f(x) = a(x-h)^2 + k$
The **vertex** $(h,k)$ is the lowest (or highest) point of the parabola, the most important point of the graph.

| | |
|---|---|
| $f(x) = x^2 + k$ | vertical shift: shift *k* units up (*k* positive) or shift *k* units down (*k* negative) |
| $f(x) = (x-h)^2$ | horizontal shift *h* units right (subtracted *h*) or shift *h* units left (added *h*)<br>CAUTION: The signs and directions are backward from the *k* signs! |
| $f(x) = (x-h)^2 + k$ | both up/down and left/right shifts |
| $f(x) = ax^2$ | change the shape (narrower $\|a\| > 1$ or wider $\|a\| < 1$)<br>and/or<br>change direction (upward opening if $a > 0$ or downward opening if $a < 0$) |
| $f(x) = ax^2 + k$ | shape and/or direction AND vertical shift |
| $f(x) = a(x-h)^2$ | shape and/or direction AND horizontal shift |
| $f(x) = a(x-h)^2 + k$ | shape and/or direction AND vertical shift AND horizontal shift |

The **axis of symmetry** is a vertical line through the vertex which divides the parabola in two mirror images. Its equation is $x = h$.

The ***x*-intercepts** are points where the parabola crosses the *x*-axis. To find them

By algebra: set $y = 0$, meaning replace the $f(x)$ by 0. Solve the resulting quadratic equation by square root property (if $f(x) = a(x-h)^2 + k$) or quadratic formula or factoring (if $f(x) = ax^2 + bx + c$).

By approximating on GC: use 2nd TRACE = CALC, option 2 (Zero) as many times as there are zeros.

Note: A quadratic function can have

- two *x*-intercepts (two real solutions to the quadratic equation, $D = b^2 - 4ac > 0$),
- one *x*-intercept (one real solution to the quadratic equation, $D = b^2 - 4ac = 0$), or
- no *x*-intercepts (two complex solutions to the quadratic equation, $D = b^2 - 4ac < 0$)

The ***y*-intercept** is the point where the parabola crosses the y-axis. To find it

By algebra: set $x = 0$, meaning find $f(0)$.

Every quadratic function has one *y*-intercept.

Note: Later, we'll also have the form $f(x) = ax^2 + bx + c$

*a* is the same as in $f(x) = a(x-h)^2 + k$, but *h* and *k* are related to *b* and *c* by the **vertex formula**:

$$h = \frac{-b}{2a}$$
$$k = f(h)$$

To change $f(x) = a(x-h)^2 + k$ to $f(x) = ax^2 + bx + c$, FOIL, distribute, and combine.

To change $f(x) = ax^2 + bx + c$ to $f(x) = a(x-h)^2 + k$, complete the square or plug in results from the vertex formula.

**Practice and Examples**

Using $f(x) = 2(x-1)^2 + 3$

1) Simplify.
2) Graph the two expressions (the given function and your result) in GC. What do you observe?
3) Identify the characteristics of the graph.
    a. What is its shape?
    b. Where is its vertex?
    c. What is the value of a and what does it mean about the direction of the graph?
4) Find the vertex of using GC. How does this answer relate to the equation?
5) Find the equation of the axis of symmetry.
6) Compare add each graph to GC, identify the effect of changes in the value of a.
    a. $f(x) = (x-1)^2 + 3$
    b. $f(x) = \frac{1}{2}(x-1)^2 + 3$
    c. $f(x) = -(x-1)^2 + 3$
    d. $f(x) = -4(x-1)^2 + 3$
    e. Summarize what the value of a tells us about the graph.
7) Find all intercepts.
    a. Use $f(x) = 2(x-1)^2 + 3$
    b. Use simplified result.
    c. Which form makes the algebra easier when finding the x-intercept? Y-intercept?
    d. What is the difference between the x-coordinate of the vertex and the x-intercept?
8) Graph $f(x) = 2(x-1)^2 + 3$ on paper.

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$f(x) = 2(x-1)^2 + 3$

① simplify

| | |
|---|---|
| $f(x) = 2(x-1)(x-1) + 3$ | exponent before mult |
| $= 2(x^2 - x - x + 1) + 3$ | FoIL |
| $= 2(x^2 - 2x + 1) + 3$ | Combine like terms |
| $= 2x^2 - 4x + 2 + 3$ | distribute |
| $= 2x^2 - 4x + 5$ | combine |

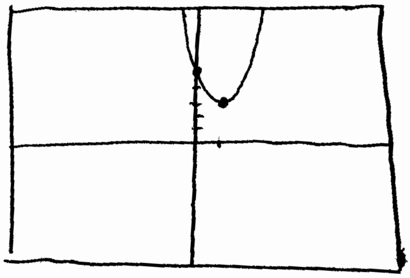
② Graph both expressions in GC.

$\boxed{Y=}$

$Y_1 = 2(x-1) \quad + 3$

$Y_2 = 2x^2 - 4x + 5$

$\boxed{\text{ZOOM}}$ 6. Zoom Standard



The two expressions give the same graph.
Simplifying does not change the graph.

③ Identify the characteristics of the graph.

a) What is its shape?

This U-shape is called a <u>parabola</u>.
All quadratic functions $f(x) = ax^2 + bx + c$ are parabolas.

b) Where is its vertex?

The vertex is the lowest point on the parabola.

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or, if the parabola opens downward, the vertex is the highest point on the parabola

When drawing the graph of a parabola, the vertex is the most important point to graph correctly.

c) What is the value of a and what does it mean about the direction of the graph?

$ax^2 + bx + c$

$2x^2 - 4x + 5$

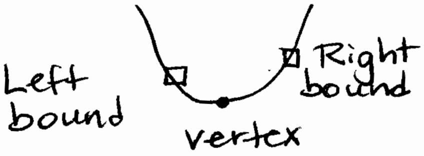
a=2 is positive, so the parabola opens upward

④ Find the vertex using GC.
How does this answer relate to the equation?

To find the vertex coordinates (or approximate ones):

CALC
2nd TRACE
3. Minimum

Left bound?
Right bound?
Guess?



Place cursor to left of vertex and press enter.

Use ▶ to move past vertex (to right of vertex) and press $\boxed{\text{enter}}$.

Press $\boxed{\text{enter}}$ a third time for the (Guess?).

minimum

$X=1 \quad Y=3$

$\boxed{\text{The coordinates of the vertex are } (1,3).}$

We notice the original function has (1,3) in it:

$$f(x) = 2(x \underset{\substack{\uparrow \\ \text{opposite} \\ \text{of the} \\ \text{x-coordinate.}}}{-} 1)^2 + \underset{\substack{\uparrow \\ \text{y-coordinate} \\ \text{of vertex}}}{3}$$

We can use this structure to "read" the vertex from the equation.

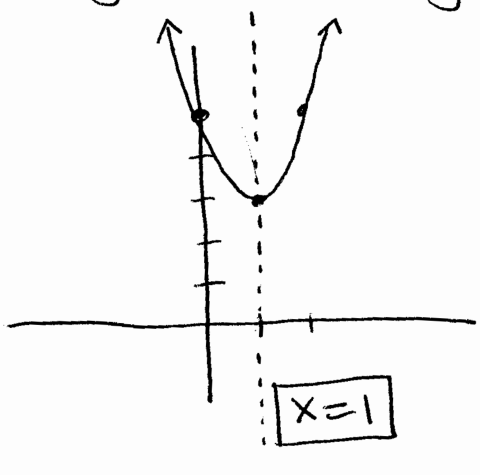
$$\boxed{\begin{array}{l} f(x) = a(x-h)^2 + k \\ \text{vertex is } (h,k) \end{array}}$$

outside ( ) ⇒ keep the sign

inside ( ) ⇒ use the opposite sign.

⑤ Find the equation of the axis of symmetry.

Parabolas are always symmetric. This means that the left half of the graph is a mirror image of the right half of the graph.



If we draw a line which divides the graph into these two halves, we have drawn the axis of symmetry.

This vertical line goes through the vertex and its equation is $x=1$.

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⑥ Compare by adding each graph to GC and identify the effect of changes in the value of a.

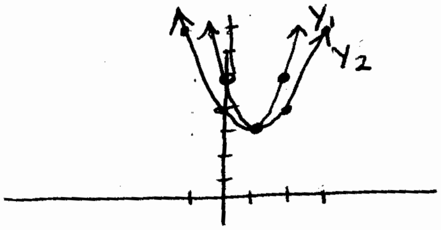
a) $f(x) = \overset{\uparrow}{\phantom{1}}(x-1)^2 + 3$

a=1 now, instead of a=2.

$\boxed{Y=}$

$Y_1 =$ leave the same

$Y_2 = (x-1)^2+3$



$Y_2$ is wider than $y_1$.

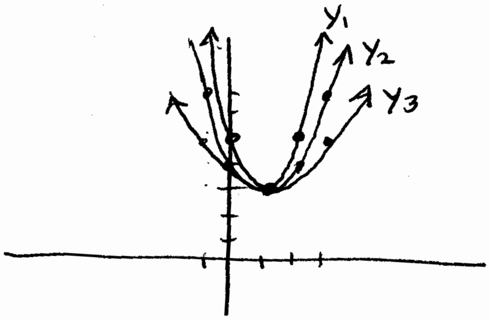
a=1 is less than a=2.

b) $f(x) = \frac{1}{2}(x-1)^2 + 3$

$\boxed{Y=}$ $Y_1 =$ leave the same

$Y_2 =$ leave the same

$Y_3 = .5(x-1)^2+3$



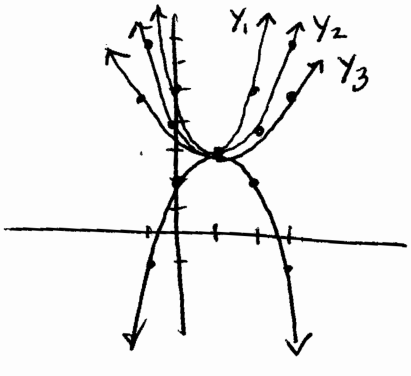
$Y_3$ is wider than both $y_1$ and $y_2$.

$a=\frac{1}{2}$ is less than both a=1 and a=2.

c) $f(x) = -(x-1)^2+3$

$\boxed{Y=}$ leave $Y_1, Y_2, Y_3$ the same

$Y_4 = -(x-1)^2+3$



$Y_4$ opens downward because a=−1 is negative.

$Y_4$ is the same width as $Y_2$, with a=1.

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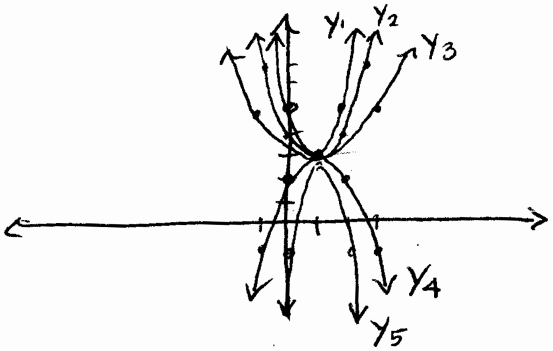
d) $f(x) = -4(x-1)^2 + 3$

$\boxed{Y=}$ leave $Y_1, Y_2, Y_3, Y_4$ the same.

$Y_5 = -4(x-1)^2 + 3$



$Y_5$ opens downward because $a = -4$ is negative.

$Y_5$ is narrower than $Y_4$ ....

but $-4$ is less than $-1$.

When deciding the width of the parabola, we use the absolute value of $a$, so we consider only the magnitude of the number, not its sign.

$(a=-4)$ $|-4| = 4$ is greater than 1

so the parabola is narrower

than the parabola $|-1| = 1$. $(a=-1)$

e) Summary:

if $|a| = 1$ we have the "basic" or "standard" parabola

if $|a| > 1$ we have a narrower parabola

if $|a| < 1$ we have a wider parabola.

⑦ Find all intercepts.

This means "find the x-intercepts" <u>and</u> "find the y-intercepts.

a) $f(x) = 2(x-1)^2 + 3$

x intercept: set $y=0$ (or replace $f(x)$ by 0)

$0 = 2(x-1)^2 + 3$ isolate the square

$-3 = 2(x-1)^2$

$\frac{-3}{2} = (x-1)^2$

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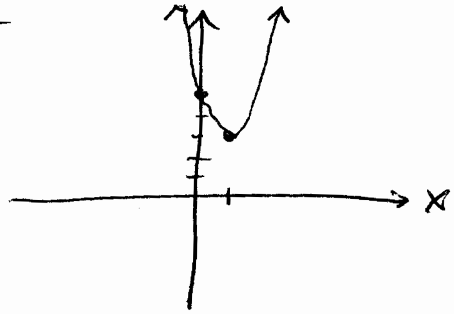
$$\pm\sqrt{\frac{-3}{2}} = x-1$$

square root property



$$1 \pm \frac{i\sqrt{3}}{\sqrt{2}} = x$$

$$x = 1 \pm \frac{i\sqrt{6}}{2}$$

imaginary x-intercepts.
(of course! this graph does not cross the x-axis!)

y-intercept: set $x=0$.

$$\begin{aligned} f(0) &= 2(0-1)^2+3 \\ &= 2(-1)^2+3 \\ &= 2(1)+3 \\ &= 2+3 \\ &= 5 \end{aligned}$$

order of operations:
parentheses first
then exponent
then multiply
then add

$\boxed{\text{y intercept } (0,5)}$

b) Use simplified result, $f(x) = 2x^2-4x+5$

x-int: set $f(x)=0$ or $y=0$

$$0 = 2x^2-4x+5$$

quadratic! Solve by factoring, QF, or CTS...

QF

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2-4(2)(5)}}{2(2)}$$

$$x = \frac{4 \pm \sqrt{-24}}{4}$$

imaginary.

$\boxed{\text{no x-ints.}}$

CTS

$$\frac{2x^2}{2} - \frac{4x}{2} = \frac{-5}{2}$$

$$x^2-2x+1 = -\frac{5}{2}+1$$

$$(x-1)^2 = \frac{-3}{2}$$

$$x-1 = \pm\sqrt{\frac{-3}{2}}$$

imaginary.

$\boxed{\text{no x-ints}}$

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y-int : set $x=0$

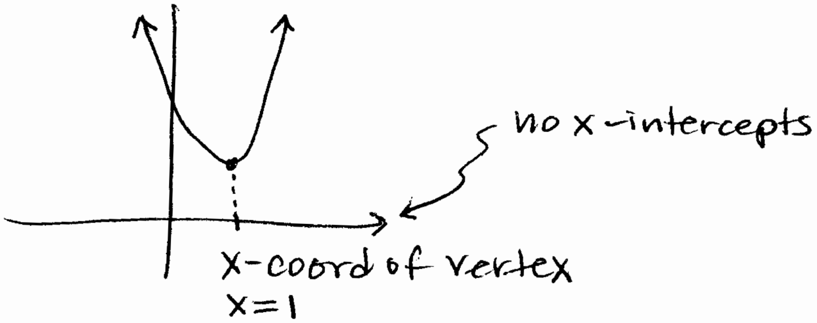
$f(0)=2(0)^2-4(0)+5$

$y=5$

$\boxed{\text{y int } (0,5)}$

c). <u>Advice:</u> The x-intercept is usually easier to find from $f(x)=a(x-h)^2+k$ but the y-intercept is usually easier to find from $f(x)=ax^2+bx+c$.

d) What is the difference between the x-coordinate of the vertex and the x-intercept?

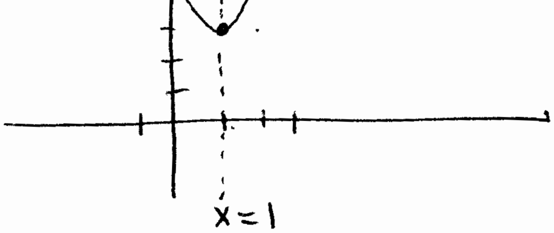


The x-intercepts are where the parabola crosses the x-axis, if it does.

The vertex is the lowest (or highest) point on the parabola; it is an ordered pair. The x-coordinate is the first number in the ordered pair.

⑧ Graph the parabola

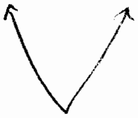
- plot the vertex
- plot at least 5 points
- make it round at vertex
- extend to edges of grid.
- in MXL, use <u>dotted</u> line for axis of symmetry



when graphing quadratic functions

1) The shape is a parabola.
This is a smooth, rounded curve without any pointy places.

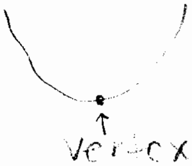

Yes.


No.


Yes


No

2) The most important point is the vertex
Always plot the vertex neatly and accurately.



3) Plot at least <u>four</u> additional points to show if the parabola is wider

or narrower

(These shape changes in 8.5 Day 2.)

4) A parabola is symmetric: the left side of the graph is the same shape as the right side
Use this to save time and check your work.

When graphing quadratic functions (continued)

5) Draw axes neatly. If scale in x direction or scale in y-direction is not 1, label scales.

6) Place axes so that 5 points will fit on the grid.

7) Extend parabola accurately to edges of grid.

8) Check x-intercepts and y-intercepts are plotted accurately.
(Compare to GC graph if x-ints or y-ints were not requested in your work.)

Remember: To find x-intercepts, set $y=0$ and solve for x.

To find y-intercept, set $x=0$ and solve for y. $(f(x))$

9) Draw axis of symmetry (if requested) using a dashed line and label it with its equation.

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## GC 30 Graph to Paper: Using the Axis of Symmetry to Graph Quadratic Functions

Objectives: Find the vertex of a quadratic function
Write the equation of the axis of symmetry for a quadratic function
Graph the axis of symmetry
Observe and use symmetry as shortcut for graphing additional points on a parabola

A Quadratic Function has a squared x, or a degree 2 term, and passes the Vertical Line Test.

Standard Form: $f(x)=a(x-h)^2+k$, where a, h, and k are constants, $a\neq 0$, and (h,k) are the coordinates of the vertex.

The axis of symmetry of a quadratic function is a vertical line through the vertex which divides the graph in two mirror halves. Its equation is $x=-\frac{b}{2a}$, same as the x-coordinate of the vertex.

**Example 1:** Find the vertex of $f(x)=(x-3)^2+2$

Using the formula $f(x)=a(x-h)^2+k$, we see that the x-coordinate can be found from (x-3), giving x = 3, and the y-coordinate can be found from +2, giving y=2. Answer: V(3,2)

**Example 2:** Write the equation of the axis of symmetry for $f(x)=(x-3)^2+2$.

Using our work from Example 1 and the fact that the axis is a vertical line through the vertex, we get the equation x=3. Answer: x=3

General Form: $f(x)=ax^2+bx+c$, where a, b, and c are constants, $a\neq 0$.

The vertex of a quadratic function can be found using the vertex formula: Find the x-coordinate first $x=-\frac{b}{2a}$, then evaluate the function at the x-value found: $y=f\left(-\frac{b}{2a}\right)$.

To write general $f(x)=ax^2+bx+c$ from standard $f(x)=a(x-h)^2+k$: FOIL and simplify.

To write standard $f(x)=a(x-h)^2+k$ from general $f(x)=ax^2+bx+c$: complete the square.

If a>0, the parabola opens upward. If a<0, the parabola opens downward.

When graphing a quadratic function, you must graph the vertex, plot at least four additional points, and neatly draw the parabola. Instructions may also require you to graph x-intercepts.

CAUTION: A parabola is a rounded shape – there should not be a point at the vertex.

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## GC 30 Graph to Paper: Using the Axis of Symmetry to Graph Quadratic Functions page 2

**Example 3:** Find the vertex of $f(x) = -2x^2 + 4x - 5$

Using the vertex formula, we find the x-coordinate using $x = -\frac{b}{2a}$, which gives $x = -\frac{4}{2(-2)} = \frac{-4}{-4} = 1$.

Plugging this into the function using $y = f\left(-\frac{b}{2a}\right)$, we get $f(1) = -2(1)^2 + 4(1) - 5 = -2 + 4 - 5 = -3$.

The vertex is (1,-3). Answer: V(1,-3)

**Example 4:** Write the equation of the axis of symmetry for $f(x) = -2x^2 + 4x - 5$.

Using our work from Example 3 and the fact that the axis is a vertical line through the vertex, we get the equation x=1. Answer: x=1

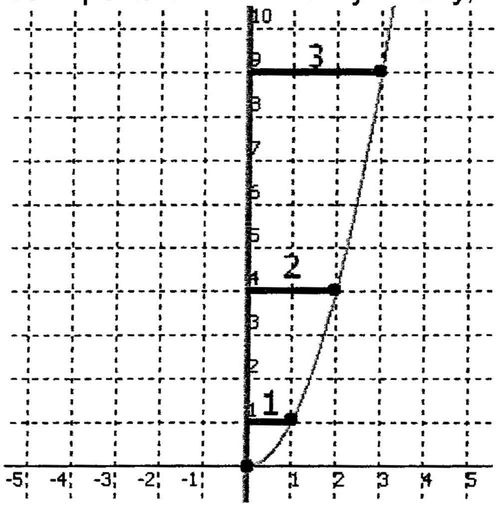
We use symmetry in graphing quadratic functions (and other parabolas), in two ways:
Forward: From the vertex, axis of symmetry, and one calculated point, graph another point on the other side of the axis of symmetry.
Backward: Check final graph and any other points (like x- or y-intercepts) for symmetry.

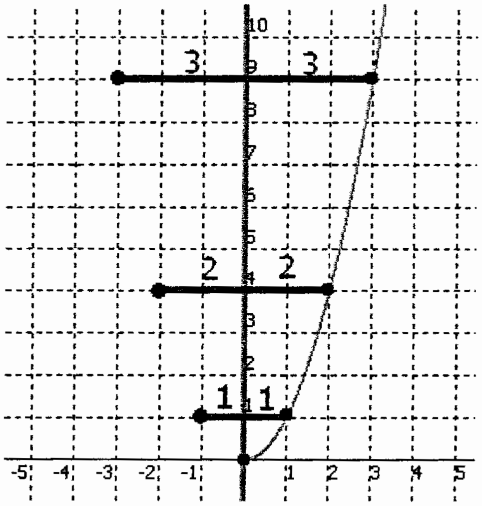
**Example 5:** Graph $f(x) = x^2$ by plotting points and using symmetry.

We know from $f(x) = a(x-h)^2 + k$ that the vertex is (0,0), so this is the first point we put in the table.

| x-value | y-value |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |
| 3 | 9 |

These points give us the right half of the parabola. To use symmetry, measure the distance from each point to the axis of symmetry, then use those same distances to draw the other half:





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## GC 30 Graph to Paper: Using the Axis of Symmetry to Graph Quadratic Functions page 3

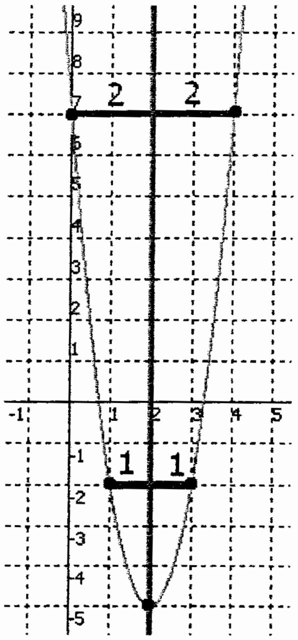
CAUTION: The axis of symmetry may be different from the y-axis. Measure distances from the axis of symmetry to the known points.

Process for graphing a quadratic function using your graphing calculator:
- Step 1: Find the vertex (using $f(x)=a(x-h)^2+k$ or the vertex formula) and axis.
- Step 2: Make a table using your GC. Make TblStart = x-coordinate of the vertex.
- Step 3: Look at the points and decide location of axes on the graph paper.
- Step 4: Determine scale for the axes. Draw and label axes.
- Step 5: Plot all the points. Draw the axis of symmetry. Use symmetry to plot the other half.
- Step 6: Neatly connect points in a rounded parabola.

**Example 6:** Neatly graph $f(x)=3(x-2)^2-5$.

Using $f(x)=a(x-h)^2+k$, the vertex is (2, -5), and the axis of symmetry is x = 2.

| x values | y values |
|---|---|
| | |
| | |
| | |
| | |



| x values | y values |
|---|---|
| 2 | -5 |
| 3 | -2 |
| 4 | 7 |

Answer

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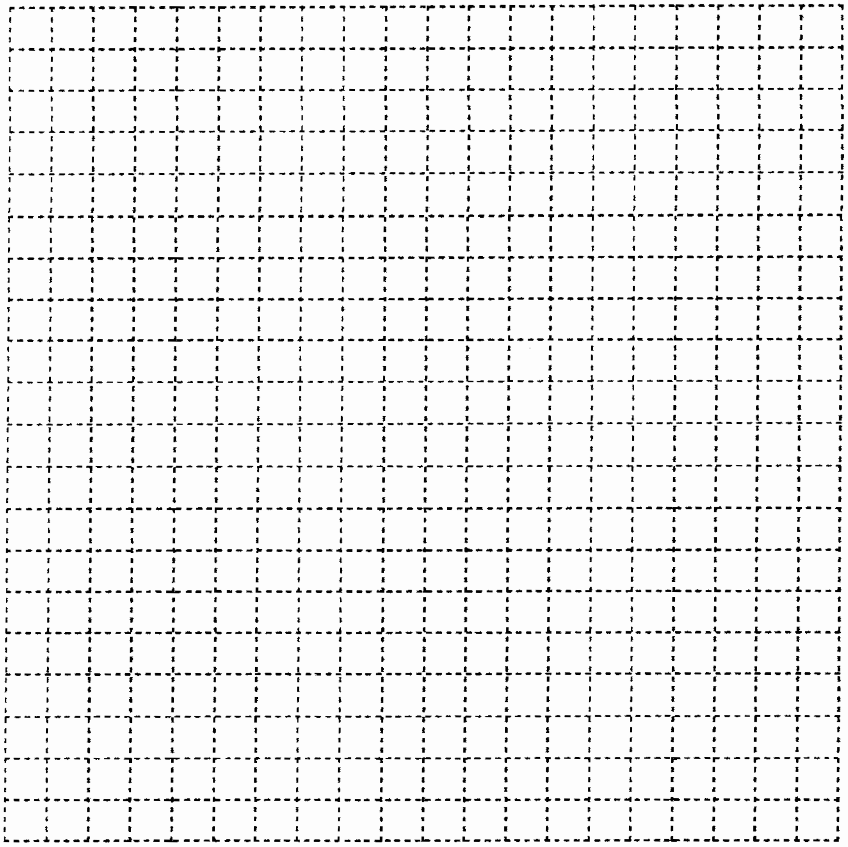
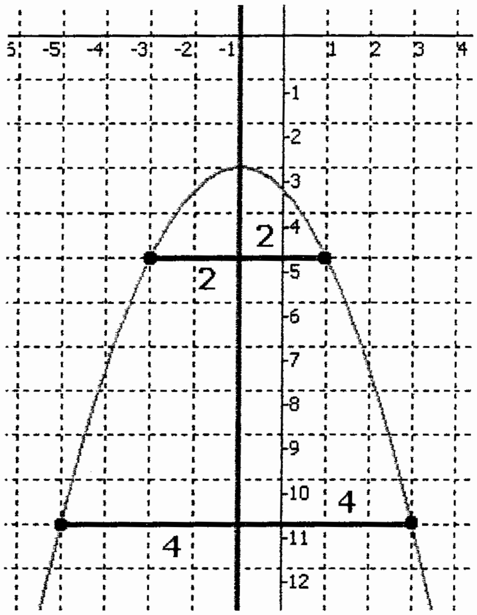
CAUTION: If quadratic functions have fractional coefficients, you must work with the fractions, just as with expressions. You cannot "clear the fractions" by multiplying by an LCD, as you can with equations.

**Example 7:** Neatly graph $f(x) = -\frac{1}{2}x^2 - x - \frac{7}{2}$.

Using the vertex formula, we get: $x = -\frac{b}{2a} = -\frac{-1}{2\left(-\frac{1}{2}\right)} = -\frac{-1}{-1} = -1$.

Evaluating $f(-1) = -\frac{1}{2}(-1)^2 - (-1) - \frac{7}{2} = -\frac{1}{2} + 1 - \frac{7}{2} = -3$. The vertex is (-1,-3).





| x values | y values |
|---|---|
| | |
| | |
| | |
| | |

Answer:

| x values | y values |
|---|---|
| -1 | -3 |
| 1 | -5 |
| 3 | -11 |

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## TI-84+ GC 31 Graph to Paper, Pattern of Squares to Graph Quadratics

Objectives: Observe the pattern of squares for quadratic functions with a=1.
Use the vertex, axis of symmetry and pattern of squares to graph quadratic functions
Use pattern of squares to graph quadratic functions when $a \neq 1$

A Quadratic Function has a $x^2$ (degree 2) term, and passes the Vertical Line Test.

General Form of a Quadratic Function: $f(x) = ax^2 + bx + c$, where a, b, and c are constants, $a \neq 0$.

Standard Form of a Quadratic Function: $f(x) = a(x-h)^2 + k$, where a, h, and k are constants, $a \neq 0$, and (h,k) are the coordinates of the vertex.

a is called the leading coefficient.
If a>0, the parabola opens upward.
If a<0, the parabola opens downward.

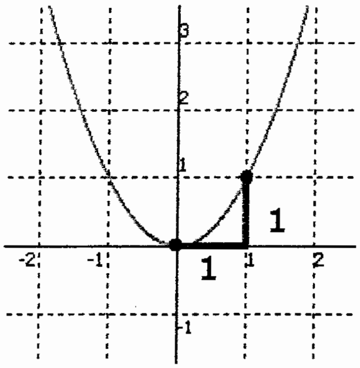
The vertex of a quadratic function can be found using the vertex formula:

First, find the x-coordinate $x = -\dfrac{b}{2a}$

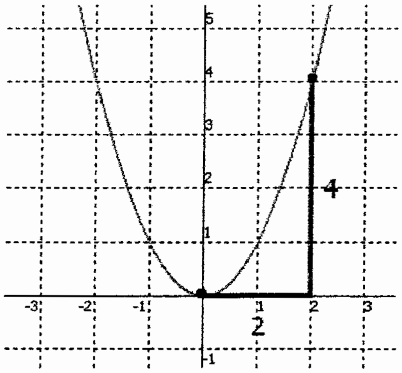
Second, evaluate the function at the x-value found: $y = f\left(-\dfrac{b}{2a}\right)$.

The axis of symmetry of a quadratic function is a vertical line through the vertex which divides the graph in two mirror halves. Its equation is $x = -\dfrac{b}{2a}$, same as the x-coordinate of the vertex.

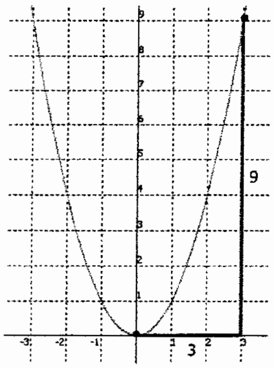
**Example 1:** Because quadratic functions have a square in their expressions, perfect squares are important in their graphs. Let's consider $f(x) = 1x^2$. In each of the graphs below, start at the vertex and move to the right. The amount we move up is the square of the amount we move right. (Notice that the leading coefficient a=1.)



$y = 1 \cdot 1^2 = 1$



$y = 1 \cdot 2^2 = 4$



$y = 1 \cdot 3^2 = 9$

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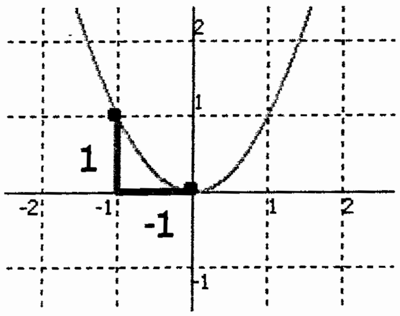
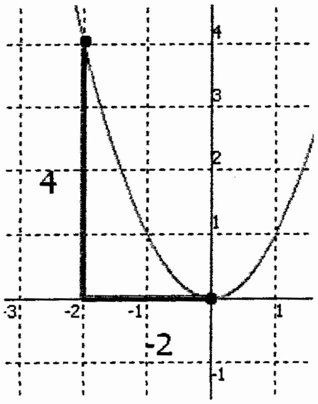
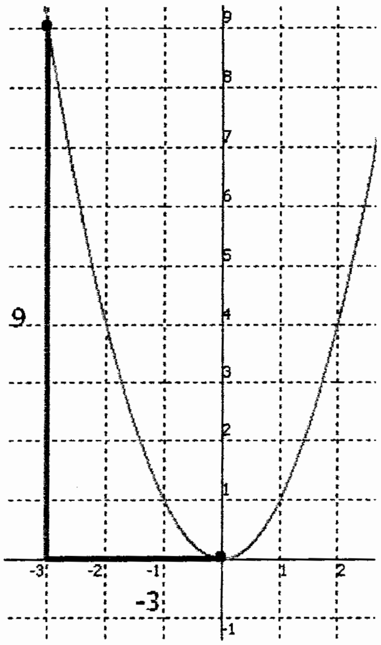
We can also move from the vertex to the left, using symmetry, and the y-coordinate is the square.



$y = 1 \cdot (-1)^2 = 1$



$y = 1 \cdot (-2)^2 = 4$

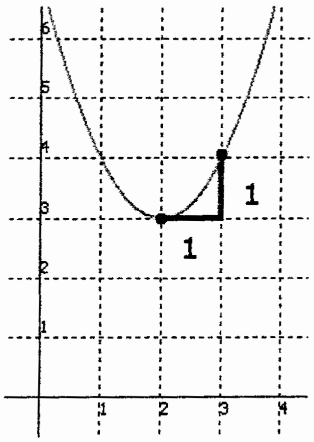


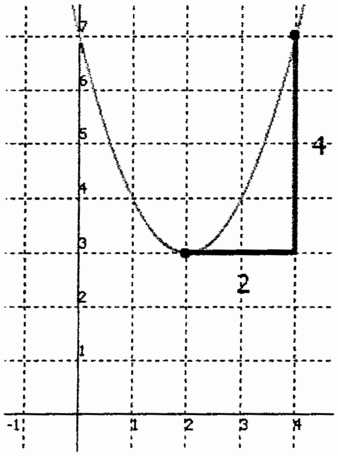
$y = 1 \cdot (-3)^2 = 9$

When the leading coefficient a=1, we can use this pattern of squares starting from any vertex.

**Example 2:** Find the vertex of $f(x) = 1(x-2)^2 + 3$. Then observe the pattern of squares in the graphs below.

Using $f(x) = a(x-h)^2 + k$, the vertex is (2,3). From the point (2, 3), we go right 1, up 1. Then left 1, up 1. Then right 2, up 4. Then left 2, up 4. And so on (left or right 3, up 9), both left and right.





Answer:

When the leading coefficient $a = -1$, we can use the same pattern of squares, but the y-coordinates are negative and go down instead of positive, going up. We begin at the vertex as before.

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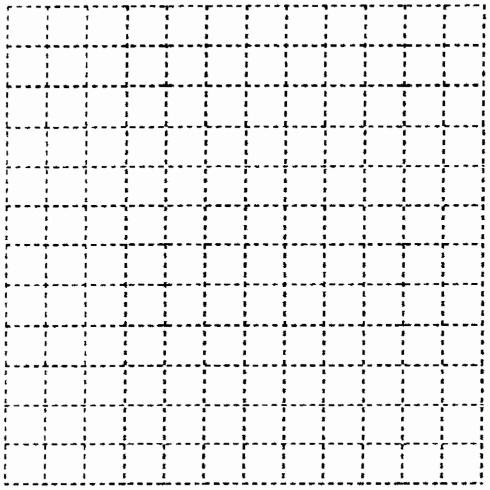
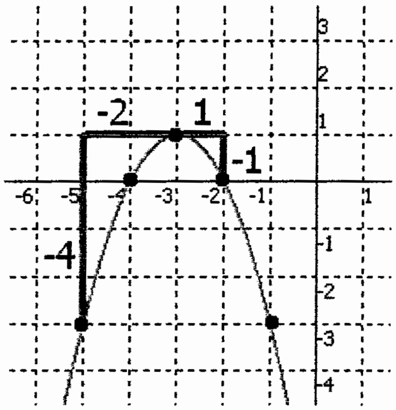
**Example 2:** Use the pattern of squares to graph $f(x) = -1(x+3)^2 + 1$

This quadratic function has leading coefficient $a = -1$ and vertex $(-3,1)$. Since a is negative, we go right 1, down 1. Then left 1, down 1. Then right 2, down 4. Then left 2, down 4.



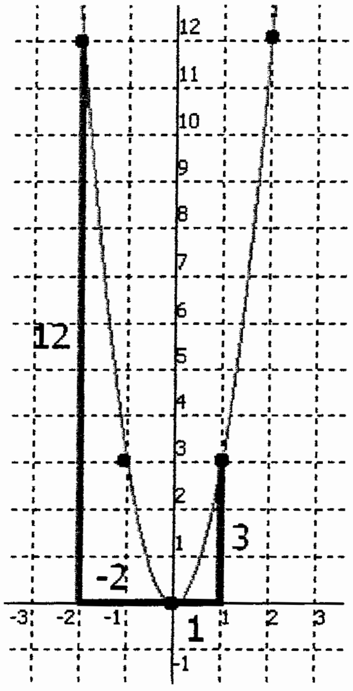
Answer:



The pattern of squares is modified using the value of a, when the leading coefficient is not 1 or -1.

CAUTION: You must use the order of operations: exponent before multiply.
Find the square, then multiply by a.

**Example 3:** Graph $f(x) = 3x^2$ using the pattern of squares.



Here, the y coordinates are $3(1)^2 = 3 \cdot 1 = 3$ and $3(-2)^2 = 3 \cdot 4 = 12$

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Practice:

1) Use the pattern of squares to graph a quadratic function with leading coefficient a = 1 whose vertex is $(-3,1))$ . Find the vertex, draw axes, then plot the vertex and points using the pattern of squares.

2) Use the pattern of squares to graph a quadratic function with leading coefficient $a = 2$ whose vertex is $(0,0))$, $f(x) = 2x^2$ Draw axes, then plot the vertex and points using the pattern of squares, modified by a=2.

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3) Use the pattern of squares to graph a quadratic function with leading coefficient $a = 2$ whose vertex is $(-1,-5))$ . Draw axes, then plot the vertex and points using the pattern of squares, modified by a=2.

4) Neatly graph $f(x) = \frac{1}{2}(x-4)^2 - 5$ using the pattern of squares. Find the vertex, draw and label axes. Find the leading coefficient and use it to modify the pattern of squares. If the y-coordinate is not an integer, skip that square and move on to the next.

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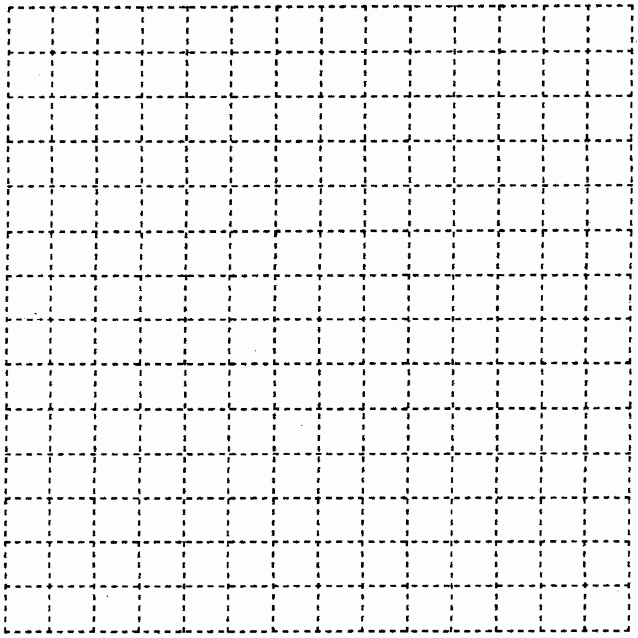
5) Neatly graph $f(x)=-\frac{1}{2}(x+4)^2+5$ using the pattern of squares. Find the vertex, draw and label axes. Find the leading coefficient and use it to modify the pattern of squares.



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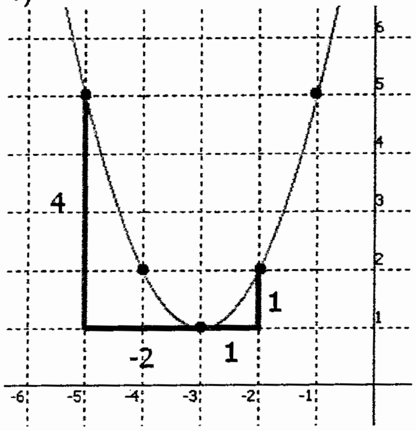
1)



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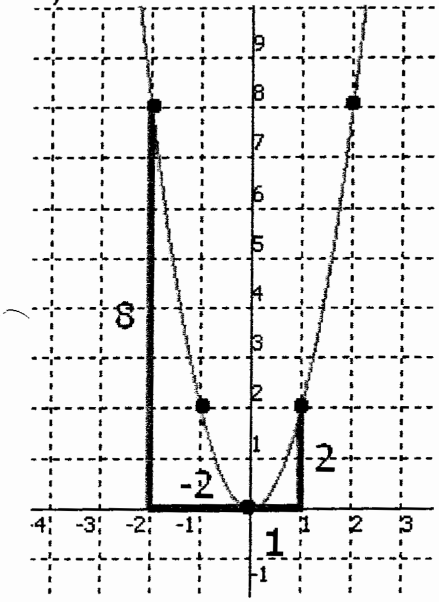

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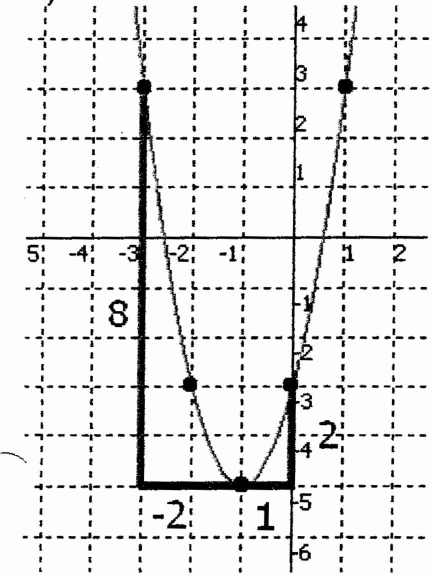

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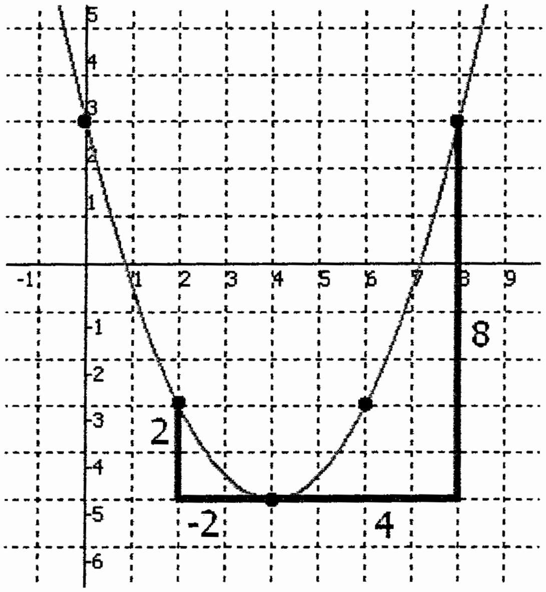

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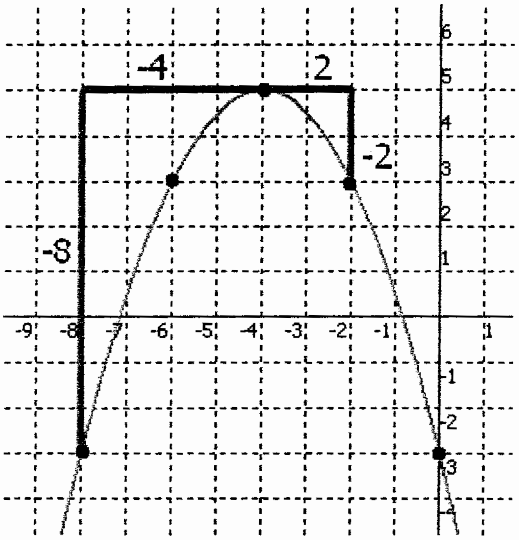

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